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Persistent Random Walks in Stationary Environment

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We study the behavior of persistent random walks (RW) on the integers in a random environment. A complete characterization of the almost sure limit behavior of these processes, including the law of large numbers, is obtained. This is done in a general situation where the environmental sequence of random variables is stationary and ergodic. Szász and Tóth obtained a central limit theorem when the ratio μ/λ, of right- and left-transpassing probabilities satisfies $\mu/\lambda \leqslant a < 1$ a.s. (for a given constant a). We consider the case where μ/λ has wider fluctuations; we shall observe that an unusual situation arises: the RW may converge a.s. to infinity even with zero drift. Then, we obtain nonclassical limiting distributions for the RW. Proofs are based on the introduction of suitable branching processes in order to count the steps performed by the RW.

KEY WORDS: Persistent random walks; random environment; branching processes.

1. INTRODUCTION

Let $\{(\lambda_j, \mu_j);\ j \in \mathbb{Z}\}$ be a sequence of stationary ergodic random variables taking their values in $]0, 1[^2$. Given a realization $\mathscr{E}$ of this sequence, let $\{X_n\}_{n \in \mathbb{N}}$ be a homogeneous Markov chain of order 2, with state space $\mathbb{Z}$ and transition probabilities:

$$\begin{aligned}
P(X_{n+1} = j+1 \mid X_{n-1} = j-1,\ X_n = j,\ \mathscr{E}) &= \lambda_j \\
P(X_{n+1} = j-1 \mid X_{n-1} = j-1,\ X_n = j,\ \mathscr{E}) &= 1 - \lambda_j \\
P(X_{n+1} = j-1 \mid X_{n-1} = j+1,\ X_n = j,\ \mathscr{E}) &= \mu_j \\
P(X_{n+1} = j+1 \mid X_{n-1} = j+1,\ X_n = j,\ \mathscr{E}) &= 1 - \mu_j
\end{aligned} \tag{1.1}$$

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When $\mathscr{E}$ is random, the $\{X_n\}$-process is not Markovian and has two levels of "stochasticity": the first one is called *environment* and is generated by the realizations of the sequence $\{(\lambda_j, \mu_j)\}$; the second one corresponds to the trajectories of a "persistent random walk" on $\mathbb{Z}$ whose behavior is given by (1.1). Thus, $\{X_n\}$ will be referred to as *Persistent Random Walk in Stationary Environment* (PRWSE).

The properties of the analogous process where $\mu_j = 1 - \lambda_j$ (called Random Walk in Random Environment) differ considerably from the properties of usual random walks and have been studied by many authors.[14, 9, 13, 11]

The present model has been introduced by Szász and Tóth in ref. 12, where they gave some applications to physical models such as "the random collision model;" "the stochastic Lorentz Gas." λ_j and μ_j were interpreted as the left- and right-transpassing probabilities characterizing a random scatterer placed on the site j ($j \in \mathbb{Z}$). The term "Persistent" was used to underline the persistence of the motion velocity (as a Markov chain) of the particle but not of the position, X_n. These authors studied the asymptotic behavior of $\{X_n\}$ in two cases:

— (S) $\mu_j = \lambda_j$ a.s., called the symmetric case, and

— (PD) $(\mu_j/\lambda_j) \leqslant a < 1$ (a is a given constant), called the positive drift case.

In the situation where the sequence $\{(\lambda_j, \mu_j)\}$ is i.i.d., they showed that $\{X_n\}$, when normed in the standard way, by $\sqrt{n}$, converges in distribution to a gaussian law. However their study left some interesting questions unresolved such as: In which cases X_n converges to $+\infty$, $-\infty$? Is the law of large numbers satisfied? Except the situation (S) and (PD), is there a limit law for X_n? If yes, does the limit law remain gaussian?

The aim of this paper is to answer the above questions. Firstly, we provide criteria for identifying the a.s. limits of X_n (Theorem 2.1 below) and those of X_n/n (Theorem 2.2 below), in a general situation of the environmental sequence ($\{(\lambda_j, \mu_j)\}$ is only stationary and ergodic). These results can be viewed as extensions of those of ref. 14 from random walks in random environments to persistent ones. Secondly, in those cases where $X_n \to +\infty$, we give a nonclassical distribution limit theorem (Theorem 2.3 and 6.1), which complements the central limit theorem of ref. 12.

This paper is organised as follows. A mathematical construction of the model and precise statements of our main results are given in the next section. In Section 3 we give a complete characterization of the limit bahavior of $\{X_n\}$ in terms of the environment. Section 4 is devoted to the study of an auxiliary Markov chain that describes "the environment as seen from the position of the random walk." The object of this is to derive some ergodic properties for $\{X_n\}$, in order to prove the law of large numbers. Some of the obtained results may be of independent interest: we give a construction of invariant measures (for the auxiliary Markov chain) that dominate the initial probability measure (which is not invariant). In Section 5, we will prove the law of large numbers. We find that one may have $X_n \to +\infty$ a.s. but $(X_n/n) \to 0$ a.s. as well. This unusual situation, which does not appear in Szász and Tóth's paper, is investigated in Section 6 in which we prove that, when suitably normed, the PRWSE converges to a non-gaussian limit distribution. This result holds when the environment is i.i.d. and $X_n \to +\infty$. It generalises the results of Kesten, Kozlov, and Spitzer[9] from random walks in random environment to persistent ones, thus extending Szász and Tóth's[12] discussion by asserting that the limiting distribution of a PRWSE may be non-gaussian.

Our main tool is based on the introduction of suitable branching processes which count the steps performed by the random walk.

2. NOTATION AND MAIN RESULTS

We begin with a mathematical construction of the model. Let $(E, \mathscr{F}, \theta, \pi)$ be an ergodic dynamical system, where θ is an invertible transformation of E and π a θ-invariant probability measure. This space is called the space of environments. Let λ and μ be two measurable functions on E with values in $]0, 1[$.

— For a fixed environment $\mathscr{E} \in E$, let $\theta^k \mathscr{E}$ denote the kth translate of $\mathscr{E}$ and $\lambda_k(\mathscr{E}) = \lambda(\theta^k \mathscr{E})$, $\mu_k(\mathscr{E}) = \mu(\theta^k \mathscr{E})$. Then, we consider on the space $\Omega = \{-1, 1\} \times \mathbb{Z}$ a Markov chain $\{(Y_n, X_n)\}$, which will have the following interpretation. Y_n is the nth jump of the random walker, X_n its position after this jump. The transition operator of this Markov chain is given by

$$Q^{\mathscr{E}}\psi(1, j) = \lambda_j \psi(1, j+1) + (1-\lambda_j)\, \psi(-1, j-1)$$

$$Q^{\mathscr{E}}\psi(-1, j) = (1-\mu_j)\, \psi(1, j+1) + \mu_j \psi(-1, j-1)$$

for ψ a bounded measurable function on Ω.

Let us denote by $P_j^{\mathscr{E}, i}$ the law of this Markov Chain on $(\Omega^{\mathbb{N}}, \mathscr{T})$, given $(Y_0, X_0) = (i, j)$, where $\mathscr{T}$ is the σ-algebra generated by the cylinder sets, $i \in \{-1, +1\}$ and $j \in \mathbb{Z}$.

— Now, when $\mathscr{E}$ is chosen at random according to the probability π, our process, $\{(Y_n, X_n)\}$, evolves on the space $(E \times \Omega^{\mathbb{N}}, \mathscr{F} \otimes \mathscr{T})$, equipped with the family of probabilities $\{P_j^{\pi, i}\}_{i=-1, +1; j \in \mathbb{Z}}$, where $P_j^{\pi, i}$ is defined by the formula

$$P_j^{\pi, i}(A \times B) = \int_A P_j^{\mathscr{E}, i}(B) \, d\pi(\mathscr{E}); \qquad A \in \mathscr{F}, \quad B \in \mathscr{T} \tag{2.1}$$

Throughout this paper we assume that $X_0 = 0$; we are interested by the study of $\{X_n\}$ under the probability $P_0^{\pi, i}$; $i = -1, +1$.

Further notation that will be used is as follows

$\prec . \succ_\pi$ will denote the average with respect to π

$E_j^{\mathscr{E}, i}$, $E_j^{\pi, i}$ will denote respectively the expectation with respect to $P_j^{\mathscr{E}, i}$, $P_j^{\pi, i}$

$T_n = \inf\{k \geqslant 0 : X_k = n\}$; $n \in \mathbb{Z}$ the times the random walk hits each integer.

In the sequel we assume that

$$\prec \log \lambda_0 \succ_\pi > -\infty \qquad \text{and} \qquad \prec \log \mu_0 \succ_\pi > -\infty \tag{2.2}$$

which in particular insures that $\prec |\log(\mu_0/\lambda_0)| \succ_\pi < +\infty$.

Theorem 2.1. (Proved in Section 3).

(i) If $\prec \log(\mu_0/\lambda_0) \succ_\pi < 0$, then for π-almost all $\mathscr{E}$, $i = -1, +1$,

$$\lim_{n \to +\infty} X_n = +\infty \qquad P_0^{\mathscr{E}, i}\text{-a.s.}$$

(ii) If $\prec \log(\mu_0/\lambda_0) \succ_\pi > 0$, then for π-almost all $\mathscr{E}$, $i = -1, +1$,

$$\lim_{n \to +\infty} X_n = -\infty \qquad P_0^{\mathscr{E}, i}\text{-a.s.}$$

(iii) If $\prec \log(\mu_0/\lambda_0) \succ_\pi = 0$, then for π-almost all $\mathscr{E}$, $i = -1, +1$,

$$\liminf_{n \to +\infty} X_n = -\infty < \limsup_{n \to +\infty} X_n = +\infty \qquad P_0^{\mathscr{E}, i}\text{-a.s.}$$

The theorem describes the a.s. recurrence-transience properties of the PRWSE. Analogous criteria for random walks in random environments (when $\mu_j = 1 - \lambda_j$) have been established in ref. 14.

The key to prove the theorem is based on a relation between persistent random walks and branching processes.

Let $U_j = \#\{k : 0 \leqslant k < T_1, X_k = j, X_{k+1} = j-1\}$; $j \leqslant 0$. We will show that, when an environment $\mathscr{E}$ is fixed, under probability $P_0^{\mathscr{E}, i}$, $\{U_j, j \leqslant 0\}$ is a inhomogeneous branching process (see refs. 3, 1, and 2 for the definitions). This kind of relation first has been found by Harris.[7] By giving an estimate to $P_0^{\mathscr{E}, i}(T_1 < +\infty)$, it turns out that $\limsup_{n \to +\infty} X_n = -\infty$ a.s. if and only if the branching process almost surely extincts; otherwise, $\limsup_{n \to +\infty} X_n = +\infty$ a.s.

The next theorem provides a law of large numbers for the PRWSE; it also gives a corresponding result for the hitting times T_n.

Further notation that will be used throughout the paper is as follows.

$$m_j = \frac{\mu_j}{\lambda_j}; \qquad r_j = \frac{1-\lambda_j}{\lambda_j}; \qquad s_j = \frac{1-\mu_j}{\mu_j}$$

$$S = S(\mathscr{E}) = \sum_{k=0}^{+\infty} m_0 . m_1 \cdots m_{k-1} . r_k$$

$$F = F(\mathscr{E}) = \sum_{k=0}^{+\infty} s_{-k} . m_{-k+1}^{-1} \cdots m_{-1}^{-1} . m_0^{-1}$$

where empty products equal 1.

Theorem 2.2. (Proved in Section 5).

(i) If $\prec S \succ_\pi < +\infty$, then

$$\lim_{n \to +\infty} \frac{X_n}{n} = v \qquad \text{and} \qquad \lim_{n \to +\infty} \frac{T_n}{n} = v^{-1} \quad \text{w.p.1} \qquad [P_0^{\pi, i}; i = -1, +1]$$

where $v = (1 + 2 \prec S \succ_\pi)^{-1}$.

(ii) If $\prec F \succ_\pi < +\infty$, then

$$\lim_{n \to +\infty} \frac{X_n}{n} = -v' \quad \text{and} \quad \lim_{n \to +\infty} \frac{T_{-n}}{n} = v'^{-1} \quad \text{w.p.1} \qquad [P_0^{\pi, i}; i = -1, +1]$$

where $v' = (1 + 2 \prec F \succ_\pi)^{-1}$.

(iii) If $\prec S \succ_\pi = +\infty$ and $\prec F \succ_\pi = +\infty$, then we have

$$\lim_{n \to +\infty} \frac{X_n}{n} = 0 \quad \text{w.p.1} \qquad [P_0^{\pi, i}; i = -1, +1]$$

and

$$\lim_{n\to+\infty} \frac{T_n}{n} = \lim_{n\to+\infty} \frac{T_{-n}}{n} = +\infty \quad \text{w.p.1} \qquad [P_0^{\pi, i}; i = -1, +1]$$

Remarks. (1) As we will see (Remarks 1 and 2 in Section 5), the conditions concerning the random environment in (i), (ii), (iii) of Theorem 2.2, are mutually exclusive and cover all possible cases. In (iii), each (but only one) of the following cases is possible:

— $\lim_{n\to+\infty} X_n = +\infty$ a.s.

— $\lim_{n\to+\infty} X_n = -\infty$ a.s.

— $-\infty = \liminf_{n\to+\infty} X_n < \limsup_{n\to+\infty} X_n = +\infty$ a.s.

(2) The conditions in (i)–(ii) of Theorem 2.2 mean that the ratio μ_0/λ_0, of right- and left-transpassing probabilities, has "small" fluctuations around the value 1. When the environmental sequence $\{(\lambda_j, \mu_j)\}$ is i.i.d. (i.e., when the space of environments is a product space with shift θ and (μ, λ) depends only on one coordinate), (i), (ii), (iii) of Theorem 2.2 respectively correspond to

$$\prec m_0 \succ_\pi < 1, \qquad \prec m_0^{-1} \succ_\pi < 1, \qquad \prec m_0 \succ_\pi^{-1} \leqslant 1 \leqslant \prec m_0^{-1} \succ_\pi$$

while $v^{-1} = 1 + 2(\prec r_0 \succ_\pi / 1 - \prec m_0 \succ_\pi)$ and $v'^{-1} = 1 + 2(\prec s_0 \succ_\pi / 1 - \prec m_0^{-1} \succ_\pi)$.

By Remark 2 above, one can see that the theorem extends Solomon's[14] law of large numbers from random walks in i.i.d. environments to persistent ones (in a more general context of environments). However, note that in i.i.d. environments, one can get the result very quickly because one can represent T_n, the time the process hits integer n, as a sum of mixing r.v.'s.

In our context, the theorem is proved by showing that in some suitable probability space the jumps of the PRWSE are stationary and ergodic. This will be done by coupling the jumps with the so-called "environment seen from the position of the random walk" (see Section 4).

While comparing the results of Theorem 2.1 to those of Theorem 2.2, we find an unusual phenomenon: we may have $X_n \to +\infty$ a.s. but $(X_n/n) \to 0$ a.s. as well. When the environmental sequence is i.i.d., according to Remark 2, the above situation occurs when

$$\prec \log \frac{\mu_0}{\lambda_0} \succ_\pi < 0 \qquad \text{but} \qquad \prec \frac{\mu_0}{\lambda_0} \succ_\pi \geqslant 1$$

Then it becomes interesting to study the existence of norming constants a_n such that X_n/a_n converges to a non-degenerate limit distribution. If we simplify the model by putting $\mu_j = 1 - \lambda_j$ a.s., the limiting distribution problem has been completely resolved by Kesten *et al.*[9] (the model then is called Random Walk In Random Environment). These authors showed in this case that X_n/n^κ converges in distribution to a non-gaussian limit, for a suitable real κ $(0 < \kappa < 1)$.

Herein, we will show that Kesten, Kozlov and Spitzer's techniques can be used succesfully to find limiting distributions for PRWSE. The main condition that we require on the environments is of "fluctuation type;" namely, $\log(\mu_0/\lambda_0)$ will be supposed to have a non-arithmetic distribution.

We give below our precise result only in the case where $X_n \to +\infty$ a.s. but $(X_n/n) \to 0$ a.s. The general result (Theorem 6.1), which also gives the limit distribution for T_n and X_n even when X_n/n has a positive limit, will be found in Section 6.

Theorem 2.3. (Part of Theorem 6.1, proved in Section 6). Assume that $\{(\lambda_j, \mu_j);\, j \in \mathbb{Z}\}$ are i.i.d. random variables satisfying (2.2) and such that

$$\prec \log \frac{\mu_0}{\lambda_0} \succ_\pi < 0, \qquad \prec \frac{\mu_0}{\lambda_0} \succ_\pi > 1, \qquad \prec \frac{1}{\lambda_0} \succ_\pi < +\infty$$

the distribution of $\log(\mu_0/\lambda_0)$ is non-arithmetic.

Then there exists $C > 0$ such that the following limit laws hold under probability $P_0^{\pi, 1}$:

$$\frac{T_n}{n^{1/\kappa}} \to L_{\kappa, C} \qquad \text{and} \qquad \frac{X_n}{n^\kappa} \to \bar{L}_{\kappa, C}$$

where

κ is the unique positive real satisfying $\prec (\mu_0/\lambda_0)^\kappa \succ_\pi = 1$ (note that $\kappa < 1$),

$L_{\kappa, C}$ is a stable distribution concentrated on $[0, +\infty[$ with index κ and characteristic function

$$\varphi_{\kappa, C}(t) = \exp\left\{ -C\, |t|^\kappa \left(1 - i \frac{t}{|t|}\, tg\left(\frac{\pi}{2} \kappa \right) \right) \right\}, \qquad t \in \mathbb{R}^*$$

$\bar{L}_{\kappa, C}$ is the distribution defined by $\bar{L}_{\kappa, C}(]-\infty, u]) = L_{\kappa, C}(]u^{-1/\kappa}, +\infty[)$

Remark. The constant C, which gives a description of the limiting distribution is complicated and is given by formula (6.10) in Section 6.

3. A RELATION BETWEEN BRANCHING PROCESSES AND PRWSE. PROOF OF THEOREM 2.1

We begin by giving estimates for $P_0^{\mathscr{E},i}(T_n < +\infty)$, where $\mathscr{E}$ is a fixed environment. We do this by using a relation between branching processes and PRWSE.

Proposition 3.1. (i) If $\prec \log(\mu/\lambda) \succ_\pi \leqslant 0$, then for π-almost all $\mathscr{E}$, $i = -1, +1$ and all $n \in \mathbb{N}$,

$$P_0^{\mathscr{E},i}(T_n < +\infty) = 1$$

(ii) If $\prec \log(\mu/\lambda) \succ_\pi > 0$, then for π-almost all $\mathscr{E}$ and $i = -1, +1$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log P_0^{\mathscr{E},i}(T_n < +\infty) \leqslant -\alpha < 0 \text{ (where } \alpha \text{ is a suitable constant)}$$

The proof of the proposition is delayed until the end of the section.

Corollary 3.2. (i)′ $\prec \log(\mu/\lambda) \succ_\pi \leqslant 0 \Rightarrow \limsup_{n \to +\infty} X_n = +\infty$ $P_0^{\mathscr{E},i}$-a.s. for π-almost all $\mathscr{E}$; $i = -1, +1$.

(ii)′ $\prec \log(\mu/\lambda) \succ_\pi > 0 \Rightarrow \lim_{n \to +\infty} X_n = -\infty$ $P_0^{\mathscr{E},i}$-a.s. for π-almost all $\mathscr{E}$; $i = -1, +1$.

Proof. (i)′ is immediate from (i) of Proposition 3.1. To show (ii)′, we have for small $\varepsilon > 0$ and sufficiently large $n \geqslant 0$, $P_0^{\mathscr{E},i}(T_n < +\infty) \leqslant e^{-(\alpha - \varepsilon) n}$ so that according to Borel–Cantelli's lemma $P_0^{\mathscr{E},i}(\limsup_{n \to +\infty} X_n = +\infty) = 0$. We show that $\limsup_{n \to +\infty} X_n = -\infty$ $P_0^{\mathscr{E},i}$-a.s.

When an environment $\mathscr{E}$ is fixed $\{(Y_n, X_n)\}$ is a Markov chain, which is irreductible since a.s. $0 < \lambda < 1$ and $0 < \mu < 1$. Then for any $k \in \mathbb{Z}$ we have

$$\begin{aligned} P_0^{\mathscr{E},i}(\limsup_{n \to +\infty} X_n \leqslant k) &= P_1^{\mathscr{E},i}(\limsup_{n \to +\infty} X_n \leqslant k) \\ &= P_1^{\mathscr{E},i}(\limsup_{n \to +\infty} (X_n - 1) \leqslant k - 1) \\ &= P_0^{\theta\mathscr{E},i}(\limsup_{n \to +\infty} X_n \leqslant k - 1) \\ &\leqslant P_0^{\theta\mathscr{E},i}(\limsup_{n \to +\infty} X_n \leqslant k) \end{aligned}$$

By using the invariance of θ, it follows that the last inequality is an equality. This proves that $P_0^{\mathscr{E},i}(\limsup_{n\to+\infty} X_n \leqslant k)$ is π-a.s. constant and does not depend on k. Now, since $\limsup_{n\to+\infty} X_n < +\infty$ a.s., it follows that $\limsup_{n\to+\infty} X_n = -\infty$ a.s. ▮

Proof of Theorem 2.1. It is a direct consequence of Corollary 3.2. (ii) is immediate from (ii)′ of Corollary 3.2.

(i) follows from (ii) by exchanging the roles of the positive and negative integers.

Let us prove (iii). If $\prec \log(\mu_0/\lambda_0) \succ_\pi = 0$, by (i)′-Corollary 3.2, $\limsup_{n\to+\infty} X_n = +\infty$ w.p.1.

By (2.2) we also have $\prec \log(\lambda_0/\mu_0) \succ_\pi = 0$. For symmetry reasons, we have $\limsup_{n\to+\infty}(-X_n) = +\infty$ w.p.1; thus $\liminf_{n\to+\infty}(X_n) = -\infty$ w.p.1. ▮

Proof of Proposition 3.1. To prove the proposition, we will use a relation between PRWSE and Branching processes. Let us fix an environment $\mathscr{E}$. The process $\{(Y_n, X_n)\}$ then is Markovian. By using the Markov property, we can write

$$\begin{aligned} P_0^{\mathscr{E},i}(T_n < +\infty) &= P_0^{\mathscr{E},i}(T_n - T_{n-1} < +\infty, T_{n-1} < +\infty)) \\ &= P_0^{\mathscr{E},i}(T_n - T_{n-1} < +\infty \mid T_{n-1} < +\infty)\, P_0^{\mathscr{E},i}(T_{n-1} < +\infty) \\ &= P_0^{\theta^{n-1}\mathscr{E},1}(T_1 < +\infty)\, P_0^{\mathscr{E},i}(T_{n-1} < +\infty) \end{aligned}$$

and by induction,

$$P_0^{\mathscr{E},i}(T_n < +\infty) = P_0^{\mathscr{E},i}(T_1 < +\infty)\, \Pi_{k=1}^{n-1} P_0^{\theta^k\mathscr{E},1}(T_1 < +\infty) \tag{3.1}$$

Then it is sufficient to study $P_0^{\mathscr{E},i}(T_1 < +\infty)$; $i = -1, +1$.

For $j < 0$ define

$$U_j = \#\{k : 0 \leqslant k < T_1, X_k = j, X_{k+1} = j-1\}$$

We have

$$T_1 = 1 + 2 \sum_{j=-\infty}^{0} U_j \tag{3.2}$$

We will see below that the U_j's are finite. Then, since they take nonnegative integer values, we have

$$P_0^{\mathscr{E},i}(T_1 < +\infty) = P_0^{\mathscr{E},i}(U_j \to 0 \text{ when } j \to -\infty) \tag{3.3}$$

The following lemma which also will be useful in the next section describes a certain aspect of the distribution of the process $(U_j;\ j \leqslant 0)$.

Put

$$Z_0 = 1, \qquad Z_n = U_{-n+1};\ n \geqslant 1$$

Lemma 3.3. ($\mathscr{E}$ is fixed). (i) Under probability $P_0^{\mathscr{E},\,i}$, $i = -1, +1$, the process $\{Z_n\}_{n \geqslant 0}$ is a branching process with time-varying law of particle reproduction given by the generating function

$$f_0(s) = \begin{cases} 1 - \mu_0 + \dfrac{\mu_0 \lambda_0 s}{1 - (1 - \lambda_0)\, s} & \text{if} \quad i = -1 \\[2ex] \dfrac{\lambda_0}{1 - (1 - \lambda_0)\, s} & \text{if} \quad i = +1, \end{cases}$$

for the particle of generation 0

$$f_n(s) = 1 - \mu_{-n} + \frac{\mu_{-n} \lambda_{-n} s}{1 - (1 - \lambda_{-n})\, s}, \qquad \text{for the particles of generation } n \geqslant 1$$

(ii) $\prec \log(\mu_0/\lambda_0) \succ_\pi \leqslant 0 \Rightarrow P_0^{\mathscr{E},\,i}(Z_n \to 0) = 1$ π-a.s., $\prec \log(\mu_0/\lambda_0) \succ_\pi > 0 \Rightarrow P_0^{\mathscr{E},\,i}(Z_n \to 0) < 1$ π-a.s.

Proof of Lemma 3.3. To prove (i), let $\mathscr{E}$ be a fixed environment. Conditionally on this, $\{(Y_n, X_n)\}$ is a Markov Chain. We have

$$\begin{aligned} &P_0^{\mathscr{E},\,+1}(U_0 = k) = \lambda_0 (1 - \lambda_0)^k; \qquad k \geqslant 0 \\ &P_0^{\mathscr{E},\,-1}(U_0 = k) = \begin{cases} 1 - \mu_0 & \text{if} \quad k = 0 \\ \mu_0 \lambda_0 (1 - \lambda_0)^{k-1} & \text{if} \quad k \geqslant 1 \end{cases} \end{aligned} \tag{3.4}$$

To describe the distribution of U_j for $j < 0$, observe that a step by $\{X_n\}$ from j to $j - 1$ has to occur between two successive steps from $j + 1$ to j. When $X_{n_0} = j$, for some n_0, then the conditional probability given $\mathscr{E}$ and $(Y_0, X_0) = (i, 0), (Y_1, X_1), \ldots, (Y_{n_0}, X_{n_0})$, of moving k times from j to $j - 1$ before the next move from j to $j + 1$ is

$$1 - \mu_j \text{ if } k = 0 \text{ and } \mu_j \lambda_j (1 - \lambda_j)^{k-1} \text{ if } k \geqslant 1$$

From this, by using the Markov Property, the distribution of U_j given $U_{j+1}, \ldots, U_0$ is that of the sum of U_{j+1} independent random variables $A_1, A_2, \ldots$, each with the distribution

$$P_0^{\mathscr{E},\,i}(A_1 = k) = \begin{cases} 1 - \mu_j & \text{if} \quad k = 0 \\ \mu_j \lambda_j (1 - \lambda_j)^{k-1} & \text{if} \quad k \geqslant 1 \end{cases} \tag{3.5}$$

Thus, under probability $P_0^{\mathscr{E},i}$, the process $\{Z_n\}_{n\geqslant 0}$ is an inhomogeneous branching process with offspring distribution given by (3.4) for the particle in the first generation and (3.5) for the particles in the $n=-j+1$th generation (see ref. 3 for the definitions concerning branching processes).

To prove (ii), using the notation $f_n = f_n^{(\mathscr{E})}$, note that when $i=-1$ we have $f_n^{(\mathscr{E})} = f_0^{(\theta^n\mathscr{E})}$; this is not true when $i=+1$. For this, the two cases will be treated separately.

Let us denote by $q_i(\mathscr{E}) = P_0^{\mathscr{E},i}(Z_n \to 0)$ the extinction probability of the branching process $\{Z_n\}$ given the environment $\mathscr{E}$.

If we let $\mathscr{E}$ to be random, when $i=-1$, $\{Z_n\}$ is an ordinary branching process in a random environment; thus, according to Athreya and Karlin,[1] $q_{-1}(\mathscr{E}) = \lim_{n\to+\infty} f_0 \circ f_1 \circ \cdots \circ f_n(0)$. Furthermore, using (2.2), we apply Corollary 1 and Theorem 3 in ref. 1 to assert that $q_{-1}(\mathscr{E})=1$ π-a.s. if $\prec \log f_0'(0) \succ_\pi = \prec \log(\mu_0/\lambda_0) \succ_\pi \leqslant 0$ and $q_{-1}(\mathscr{E})<1$ π-a.s. otherwise.

In the situation where $i=1$, one can remark that

$$q_1(\mathscr{E}) = f_0(\lim_{n\to+\infty} f_1 \circ f_2 \circ \cdots \circ f_n(0)) = f_0(q_{-1}(\theta\mathscr{E}))$$

and we get the result from the case $i=-1$ since f_0 is a generating function. ∎

Now we prove (i) of the proposition. If $\prec \log(\mu_0/\lambda_0) \succ_\pi \leqslant 0$, by (3.3) and (ii) of Lemma 3.3,

$$P_0^{\mathscr{E},i}(T_1 < +\infty) = 1 \qquad \pi\text{-a.s.}$$

Thus, $P_0^{\mathscr{E},i}(T_n < +\infty) = 1$ π-a.s. by (3.1).

To prove (ii) of the proposition, from (3.1) we have

$$\limsup_{n\to+\infty} \frac{1}{n} \log P_0^{\mathscr{E},i}(T_n < +\infty) \leqslant \limsup_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n-1} \log q_1(\theta^k\mathscr{E})$$

By (ii) of Lemma 3.3, $\prec \log(\mu_0/\lambda_0) \succ_\pi > 0$ implies $-\infty \leqslant \prec \log q_1(\mathscr{E}) \succ_\pi < 0$. Then (ii) of the proposition follows from the ergodic theorem. ∎

4. THE ENVIRONMENT AS SEEN FROM THE POSITION OF THE WALKER

We note that the law of large numbers for the PRWSE $\{X_n\}$ cannot be achieved by applying the ergodic theorem because the jumps are not stationary. The object of this section is to show that in some suitable probability space these jumps are stationary and ergodic.

An economical way to do this, is to study the following Markov Chain, which is closely related to the PRWSE:

$$V_n = (\theta^{X_n}., Y_n); \qquad n \geqslant 0$$

It has state space $E \times \{-1, +1\}$ and is obtained by coupling the jump $Y_n = X_n - X_{n-1}$ of the PRWSE, with the "environment seen by the walker after this jump."

Under the probability $P_0^{\pi, i}$; $i = -1, +1$, the initial distribution of $\{V_n\}$ is $\pi \otimes \delta_i$ and the transition operator is given by

$$K\varphi(\mathscr{E}, 1) = \varphi(\theta\mathscr{E}, 1)\, \lambda(\mathscr{E}) + \varphi(\theta^{-1}\mathscr{E}, -1)(1 - \lambda(\mathscr{E}))$$

$$K\varphi(\mathscr{E}, -1) = \varphi(\theta\mathscr{E}, 1)(1 - \mu(\mathscr{E})) + \varphi(\theta^{-1}\mathscr{E}, -1)\, \mu(\mathscr{E})$$

for φ a bounded measurable function on $E \times \{-1, +1\}$.

For ν a probability measure on $E \times \{-1, +1\}$ (that is an initial distribution of $\{V_n\}$) we define on $E \times \Omega^{\mathbb{N}}$ the probability P_0^ν by

$$P_0^\nu = \int_{E \times \{-1, +1\}} P_0^{\mathscr{E}, i}\, d\nu(\mathscr{E}, i) \tag{4.1}$$

Then, if ν is K-invariant, the Markov chain $\{V_n\}$ is stationary under probability P_0^ν. Note that $\pi \otimes \delta_i$ is not K-invariant. Theorem 4.1 below shows that $\{V_n\}$ has an invariant probability measure ν, which dominates $\pi \otimes \delta_i$.

We recall some notation that will be used

$$m_j = \frac{\mu_j}{\lambda_j}; \qquad r_j = \frac{1 - \lambda_j}{\lambda_j}; \qquad s_j = \frac{1 - \mu_j}{\mu_j}$$

$$S = S(\mathscr{E}) = \sum_{k=0}^{+\infty} m_0 . m_1 \cdots m_{k-1} . r_k$$

$$F = F(\mathscr{E}) = \sum_{k=0}^{+\infty} s_{-k} . m_{-k+1}^{-1} \cdots m_{-1}^{-1} . m_0^{-1}$$

Theorem 4.1. If $\prec S \succ_\pi < +\infty$ (respectively $\prec F \succ_\pi < +\infty$), then there exists a $\{V_n\}$-invariant probability measure ν, which dominates $\pi \otimes \delta_i$; $i = -1, 1$; ν is given by

$$\nu = v(1 + S(\mathscr{E}))\, \pi \otimes \delta_1 + vS(\theta\mathscr{E})\, \pi \otimes \delta_{-1} \tag{4.2}$$

(respectively $\nu' = v'(1 + F(\mathscr{E}))\, \pi \otimes \delta_{-1} + v'F(\theta^{-1}\mathscr{E})\, \pi \otimes \delta_1$) where v (resp. v') is a normalization constant.

Furthermore, under the probability P_0^{ν}, $\{V_n\}$ is stationary and ergodic.

An analogous formula to (4.2) is given in the case of finitely dependent environment in ref. 12 (see the remark after Theorem 2). In our context, for seek of completeness we will write the proofs.

Remark. Before proving the theorem, we note that $\prec S \succ_\pi < +\infty$ implies $X_n \to +\infty$ a.s. $[P_0^{\pi, i}, i = -1, +1]$.

To see this, note that $\prec S \succ_\pi = \sum_{k=0}^{+\infty} \prec m_{-k} \cdots m_{-1} r_0 \succ_\pi = \prec \sum_{k=0}^{+\infty} m_{-k} \cdots m_{-1} r_0 \succ_\pi$. It follows that $\prec S \succ_\pi < +\infty$ implies $\sum_{k=-\infty}^{0} m_{-k} \cdots m_{-1} < +\infty$ a.s. Now,

$$\log m_{-k} \cdots m_{-1}$$
$$= \sum_{j=-k}^{-1} \log m_j \to -\infty \text{ a.s. implies } \prec \log m_{-1} \succ_\pi < 0 \tag{4.3}$$

by Lemma 3.6 of ref. 6. Then Theorem 2.1 concludes that a.s. $X_n \to +\infty$.

Proof. We only prove the result when $\prec S \succ_\pi < +\infty$; the corresponding result when $\prec F \succ_\pi < +\infty$ being similar. By the above remark, with $P_0^{\pi, 1}$-probability 1, $\lim_{n \to +\infty} X_n = +\infty$ so that for any $n \geqslant 1$, we have $T_n < +\infty$. Then if we read the $\{V_n\}$-process at the random times T_n, we obtain $\{V_{T_n} = (\theta^n ., 1)\}$, which is a "degenerate" Markov chain with invariant probability measure $\pi \otimes \delta_1$. Then we set

$$\nu(B) = E_0^{\pi, 1}\left(\sum_{k=0}^{T_1 - 1} \mathbf{1}_B(V_k)\right);$$
$$\text{for } B \text{ a measurable subset of } E \times \{-1, +1\} \tag{4.4}$$

$\nu(B)$ is the expected number of visits to B performed by $\{V_n\}$ before time T_1.

It is not difficult to see that ν is a well-defined measure. Our first aim below is to show that ν is invariant for $\{V_n\}$. Next, we will find an explicit formula, which shows that $\pi \otimes \delta_i$; $i = -1, 1$ is absolutely continuous w.r.t. ν.

(a) Invariance of ν: Let φ be a measurable non-negative function on $E \times \{-1, +1\}$. In the sequel $\nu(\varphi)$ will denote its mean relative to the measure ν: $\nu(\varphi) = \sum_{k=0}^{+\infty} E_0^{\pi, 1}(\varphi(V_k), T_1 > k)$. Then we have

$$\nu(K\varphi) = \sum_{k=0}^{+\infty} E_0^{\pi,1}(E_0^{\pi,1}(\varphi(V_{k+1}) \mid V_k), T_1 > k)$$

$(\{V_k\}$ is a Markov chain)

$$= \sum_{k=0}^{+\infty} E_0^{\pi,1}(E_0^{\pi,1}(\varphi(V_{k+1}) \mid V_k, V_{k-1},..., V_1), T_1 > k)$$

$$= \sum_{k=0}^{+\infty} E_0^{\pi,1}(E_0^{\pi,1}(\varphi(V_{k+1}), T_1 > k \mid V_k, V_k, V_{k-1},..., V_1))$$

(because $\{T_1 > k\} = \{Y_1 \leqslant 0, Y_1 + Y_2 \leqslant 0,..., Y_1 + \cdots + Y_k \leqslant 0\}$).

$$\nu(K\varphi) = \sum_{k=0}^{+\infty} E_0^{\pi,1}(\varphi(V_{k+1}), T_1 > k)$$

$$= \sum_{k=0}^{+\infty} E_0^{\pi,1}(\varphi(V_{k+1}), T_1 = k+1) + \sum_{k=0}^{+\infty} E_0^{\pi,1}(\varphi(V_{k+1}), T_1 > k+1)$$

$$= E_0^{\pi,1}(\varphi(V_{T_1}), T_1 > 0) + \sum_{k=1}^{+\infty} E_0^{\pi,1}(\varphi(V_k), T_1 > k) = \nu(\varphi)$$

In the last equality we used that $E_0^{\pi,1}(\varphi(V_{T_1})) = \int \varphi(\theta\mathscr{E}, 1)\, d\pi(\mathscr{E}) = \int \varphi(\mathscr{E}, 1)\, d\pi(\mathscr{E}) = E_0^{\pi,1}(V_0)$.

(b) An explicit formula for ν: We have for a measurable non-negative function φ:

$$\nu(\varphi) = E_0^{\pi,1}\left(\sum_{k=0}^{T_1-1} \varphi(V_k)\right)$$

$$= E_0^{\pi,1}\left(\sum_{j=-\infty}^{0} \varphi(\theta^j\mathscr{E}, -1)\, N_j^- + \varphi(\theta^j\mathscr{E}, +1)\, N_j^+\right)$$

where $N_j^- = \#\{k : 0 \leqslant k < T_1 \text{ and } V_k = (\theta^j\mathscr{E}, -1)\}$

$N_j^+ = \#\{k : 0 \leqslant k < T_1 \text{ and } V_k = (\theta^j\mathscr{E}, +1)$

But since $T_1 < +\infty$ a.s., one can easily see that we have $P_0^{\pi,1}$-a.s $N_j^- = U_{j+1}$, $N_j^+ = U_j$ for $j < 0$; and $N_0^- = 0$, $N_0^+ = 1 + U_0$, where

$$U_j = \{k : 0 \leqslant k < T_1, X_k = j, X_{k+1} = j-1\}; \qquad j \in \mathbb{Z}^-$$

Thus, by using the definition (2.1), the branching property described in (i) of Lemma 3.3, and the fact that π is θ-invariant, we have

$$\begin{aligned}
v(\varphi) &= \int_E \varphi(\mathscr{E}, 1)(1 + E_0^{\mathscr{E}, 1}(U_0))\, d\pi(\mathscr{E}) \\
&\quad + \int_E \sum_{j=-\infty}^{-1} \varphi(\theta^j \mathscr{E}, -1)\, E_0^{\mathscr{E}, 1}(U_{j+1})\, d\pi(\mathscr{E}) \\
&\quad + \int_E \sum_{j=-\infty}^{-1} \varphi(\theta^j \mathscr{E}, 1)\, E_0^{\mathscr{E}, 1}(U_j)\, d\pi(\mathscr{E}) \\
&= \int_E \varphi(\mathscr{E}, 1)(1 + r_0)\, d\pi(\mathscr{E}) \\
&\quad + \int_E \sum_{j=-\infty}^{-1} \varphi(\theta^j \mathscr{E}, -1)(m_{j+1}.m_{j+2} \cdots m_{-1}.r_0)\, d\pi(\mathscr{E}) \\
&\quad + \int_E \sum_{j=-\infty}^{-1} \varphi(\theta^j \mathscr{E}, 1)(m_j.m_{j+1} \cdots m_{-1}.r_0)\, d\pi(\mathscr{E}) \\
&= \int_E \varphi(\mathscr{E}, 1)(1 + r_0)\, d\pi(\mathscr{E}) \\
&\quad + \sum_{j=-\infty}^{-1} \int_E \varphi(\mathscr{E}, -1)(m_1.m_2 \cdots m_{-j-1}.r_{-j})\, d\pi(\mathscr{E}) \\
&\quad + \sum_{j=-\infty}^{-1} \int_E \varphi(\mathscr{E}, 1)(m_0.m_1 \cdots m_{-j-1}.r_{-j})\, d\pi(\mathscr{E}) \\
&= \int_E (\varphi(\mathscr{E}, 1)(1 + r_0 + m_0.r_1 + m_0.m_1.r_2 + \cdots) \\
&\quad + \varphi(\mathscr{E}, -1)(r_1 + m_1.r_2 + m_1.m_2.r_3 + \cdots))\, d\pi(\mathscr{E})
\end{aligned}$$

Therefore, we can write $v(d\mathscr{E} \otimes dy) = (1 + S(\mathscr{E}))\, \pi(d\mathscr{E})\, \delta_1(dy) + S(\theta\mathscr{E})\, \pi(d\mathscr{E})\, \delta_{-1}(dy)$. Let v be the constant given by $v^{-1} = v(E \times \{-1, +1\})$. Using definition (4.4), we can write

$$v^{-1} = E_0^{\pi, 1}(T_1) = \int_E (1 + S(\mathscr{E})\, \pi(d\mathscr{E}) + \int_E S(\theta\mathscr{E})\, \pi(d\mathscr{E}) = 1 + 2 \prec S(\mathscr{E}) \succ_\pi \tag{4.5}$$

Since $\prec S(\mathscr{E}) \succ_\pi < +\infty$, we normalize v to obtain a $\{V_n\}$-invariant probability measure, which we still denote by v. The latter has the form (4.2).

The fact that $\pi \otimes \delta_i$; $i = -1, 1$, is absolutely continuous w.r.t. v is obvious since $S(\mathscr{E})$ is a positive function.

(c) Ergodicity of $\{V_n\}$: That $\{V_n\}$ is stationary under the probability P_0^ν comes from the fact that ν is K-invariant. To have ergodicity, we prove that bounded K-harmonic functions are constant ν-almost everywhere.

Let $\varphi(\mathscr{E}, i)$ be such a function. By Schwarz inequality,

$$\varphi^2 = (K\varphi)^2 \leqslant K(\varphi^2) \tag{4.6}$$

Thus $\nu(\varphi^2) = \nu((K\varphi)^2) \leqslant \nu(K(\varphi^2)) = \nu(\varphi^2)$ and then $\nu((K\varphi)^2) = \nu(K(\varphi^2))$. Using (4.6), we get for ν-almost all $(\mathscr{E}, i) \in E \times \{-1, +1\}$,

$$(K\varphi)^2 \, (\mathscr{E}, i) = K(\varphi^2)(\mathscr{E}, i)$$

This shows that φ is $K((\mathscr{E}, i), .)$-a.s. constant. But, the probability measure $K((\mathscr{E}, i), .)$ is supported by the set $\{(\theta\mathscr{E}, 1), (\theta^{-1}\mathscr{E}, -1)\}$ (because $0 < \mu(\mathscr{E}) < 1$ and $0 < \lambda(\mathscr{E}) < 1$). Therefore, for π-almost every $\mathscr{E}$, we have $\varphi((\theta\mathscr{E}, 1) = \varphi(\theta^{-1}\mathscr{E}, -1)$. Using again the harmonicity of φ we obtain

$$\varphi(\mathscr{E}, 1) = \varphi(\theta\mathscr{E}, 1) = \varphi(\theta^{-1}\mathscr{E}, -1) \qquad \text{for } \pi-\text{almost all } \mathscr{E} \in E \tag{4.7}$$

Thus $\varphi(., 1)$ is constant π-a.s. since it is θ-invariant. Finally (4.7) shows that φ is constant ν-a.s. ∎

5. THE LAW OF LARGE NUMBERS. PROOF OF THEOREM 2.2

In Section 2 we obtained the a.s. limits of X_n; herein we study those of X_n/n. We also give corresponding results for the sequence of hitting times $\{T_n\}_{n \in \mathbb{Z}}$. The proof of the law of large numbers (Theorem 2.2) will use the results of the previous section.

Remarks. Before proving Theorem 2.2, notice that

(1) The three cases in (i), (ii), (iii) of Theorem 2.2 concerning the random environment, are mutually exclusive and cover all possible cases.

(2) In case (i) (respectively (ii)), $X_n \to +\infty$ a.s. ($X_n \to -\infty$ a.s.). But in case (iii), each (but only one) of the following behaviors is possible:

— $\lim_{n \to +\infty} X_n = +\infty$ a.s.

— $\lim_{n \to +\infty} X_n = -\infty$ a.s.

— $-\infty = \liminf_{n \to +\infty} X_n < \limsup_{n \to +\infty} X_n = +\infty$ a.s.

To see this, note that

$$\prec S \succ_\pi = \prec \sum_{k=0}^{+\infty} m_{-k} \cdots m_{-1} r_0 \succ_\pi$$

and

$$\prec F \succ_\pi = \prec \sum_{k=0}^{+\infty} s_0 . m_1^{-1} \cdots m_k^{-1} \succ_\pi \tag{5.1}$$

Put

$$D = \left\{ \mathscr{E} : \sum_{n=0}^{+\infty} m_{-n} \cdots m_{-1} < +\infty \right\}; \qquad D' = \left\{ \mathscr{E} : \sum_{n=0}^{+\infty} m_1^{-1} \cdots m_n^{-1} < +\infty \right\}$$

D and D' are θ-invariant sets; therefore, each of them has probability 0 or 1. Furthermore, $\pi(D) = 1$ implies $\pi(D') = 0$ since $m_{-k} \cdots m_{-1}$ and $m_1 \cdots m_k$ have the same distribution.

To see (1), we use (5.1) to derive that $\prec S \succ_\pi < +\infty$ implies $\pi(D) = 1$ and $\pi(D') = 0$, which in turn implies $\prec F \succ_\pi = +\infty$. Similarly, $\prec F \succ_\pi < +\infty$ implies $\prec S \succ_\pi = +\infty$.

Now, using (4.3) and the ergodic theorem, we have

$$\pi(D) = 1 \text{ if and only if } \prec \log m_{-1} \succ_\pi < 0$$

Consequently, three cases are possible for the environment:

(a) $\pi(D) = 1$ and $\pi(D') = 0$ in which case $\prec \log m_0 \succ_\pi < 0$

(b) $\pi(D) = 0$ and $\pi(D') = 1$ in which case $\prec \log m_0 \succ_\pi > 0$

(c) $\pi(D) = 0 = \pi(D')$ in which case $\prec \log m_0 \succ_\pi = 0$

To see (2), (5.1) shows that $\prec S \succ_\pi < +\infty$ (respectively $\prec F \succ_\pi < +\infty$) implies (a) (resp. (b)) but if $\prec S \succ_\pi = \prec F \succ_\pi = +\infty$ then any one of the above situations could hold. Then Remark 2 follows by Theorem 2.1.

Proof of Theorem 2.2. For symmetry reasons, (ii) can be deduced from (i) by exchanging the roles of the positive and negative integers.

(i) In the light of Theorem 4.1, we will calculate the limit of X_n/n under probability P_0^ν (defined by (4.1)) where ν is given by (4.1). Since $\pi \otimes \delta_i$ is absolutely continuous w.r.t. ν we derive that $P_0^{\pi, i}$; $i = -1, +1$ is absolutely continuous w.r.t. P_0^ν; it follows that the calculated limit remains valid under $P_0^{\pi, i}$; $i = -1, +1$.

We write $X_n = \sum_{k=1}^n Y_k$ (where $Y_k = X_k - X_{k-1}$) as the sum of its increments; then using Theorem 4.1 and Birkhoff's ergodic theorem, we have

$$\lim_{n \to +\infty} \frac{X_n}{n} = E_0^\nu(Y_1)$$
$$= v\left(\int_E (1 + S(\mathscr{E}))\, d\pi(\mathscr{E}) - \int_E S(\theta\mathscr{E})\, d\pi(\mathscr{E})\right)$$
$$= v \qquad P_0^\nu\text{-a.s.}$$

The corresponding result for T_n follows very quickly by a classical argument (see e.g. pp. 7–8 of ref. 14), which shows that

$$\lim_{n \to +\infty} \frac{T_n}{n} = \lim_{n \to +\infty} \frac{n}{X_n} \qquad \text{a.s.} \tag{5.2}$$

Part (iii) cannot be proved in the same way as (i) (for the $\{V_n\}$-process introduced in Section 4, no invariant probability measure is known[2]). At first, we prove the result for T_n (the result for T_{-n} follows by a reversal argument); next, the corresponding result for X_n will follow from (5.2).

In the light of Remark 2 above, we consider two cases.

When $\limsup X_n = -\infty$, the result is obvious since in this case, a.s. $T_n = +\infty$ for sufficiently large n.

When $\limsup X_n = +\infty$, define $\tau_k = T_k - T_{k-1}$; $k \geqslant 1$. Let us consider for $A > 0$ the truncated r.v.'s

$$\tau_k^A = \tau_k \qquad \text{if} \quad \tau_k < A$$
$$= 0 \qquad \text{otherwise}$$

Recall that when an environment $\mathscr{E}$ is fixed, $\{(Y_n, X_n)\}$ is a Markov chain. By using the strong Markov property, one can see that the τ_k's are independent r.v.'s. On the other hand,

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} E_0^{\mathscr{E}, i}(\tau_k^A) = \lim_{n \to +\infty} \frac{1}{n}\left(E_0^{\mathscr{E}, i}(\tau_1^A) + \sum_{k=1}^{n-1} E_0^{\theta^k\mathscr{E}, 1}(\tau_1^A)\right)$$
$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n-1} E_0^{\theta^k\mathscr{E}, 1}(\tau_1^A)$$
$$= E_0^{\pi, 1}(\tau_1^A) \qquad \text{(by the ergodic theorem)}$$

for a set of $\mathscr{E}$'s, denote it E_1, such that $\pi(E_1) = 1$.

[2] The existence of such a probability measure is not excluded by our method.

Thus, the law of large numbers for independent r.v.'s (with arbitrary distributions) (see Theorem 3, p. 239 of ref. 4) gives for all $\mathscr{E} \in E_1$

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \tau_k^A = E(\tau_1^A) \qquad P_0^{\mathscr{E}, i}\text{-a.s.}; \quad i = -1, +1$$

But

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \tau_k \geqslant \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \tau_k^A = E_0^{\pi, 1}(\tau_1^A)$$

so that by the monotone convergence theorem (when $A \to +\infty$) $\liminf_{n \to +\infty} (1/n) \sum_{k=1}^{n} \tau_k \geqslant E^{\pi, 1}(\tau_1)$. Now $E_0^{\pi, 1}(\tau_1) = +\infty$ by (4.5); thus

$$\lim_{n \to +\infty} \frac{T_n}{n} = +\infty \qquad P_0^{\mathscr{E}, i}\text{-a.s.}; \quad i = -1, +1 \quad \blacksquare$$

6. A LIMIT LAW FOR A PRWSE

In this section we consider the case where $\lim_{n \to +\infty} X_n = +\infty$. We study the existence of constants a_n and b_n such that the r.v. $(X_n - b_n)/a_n$ converges in distribution to a non-degenerate limit. Here, we will limit our investigations to the situation where the environmental sequence $\{(\lambda_j, \mu_j)\}$ is i.i.d.

In the case where $(\mu_j/\lambda_j) \leqslant a < 1$ a.s. (this is a special case of (i)-Theorem 2.2, which tells us that X_n/n has a positive limit), Szász and Tóth[12] found that homogeneization phenomenon holds for the PRWSE (Central Limit Theorem with the standard normalization $\sqrt{n}$). Their method used the fact that T_n can be represented as a sum of exponentially mixing r.v.'s.

Herein we allow μ_j/λ_j to have wider fluctuations so that, according to Theorem 2.2 and its following remarks, one may have $X_n \to +\infty$ a.s. but $(X_n/n) \to 0$ a.s. as well. This occurs precisely (recall that the environment is i.i.d.) when

$$\prec \log \frac{\mu_0}{\lambda_0} \succ_\pi < 0 \qquad \text{but} \qquad \prec \frac{\mu_0}{\lambda_0} \succ_\pi \geqslant 1$$

and corresponds to the case where $T_1 < +\infty$ a.s. but $E_0^{\pi, i}(T_1) = +\infty$.

Such a phenomenon first has been observed by Solomon[14] for the so called Random Walks in Random Environment (RWIRE) (the RWIRE model corresponds to the situation where the left- and right-transpassing probabilities satisfy $\mu_j = 1 - \lambda_j$ a.s.), for which Kesten, Kozlov, and Spitzer[9] found nonclassical limiting distributions. They showed that in this case, for suitable κ $(0 < \kappa < 1)$, $T_n/n^{1/\kappa}$ converges to a stable distribution with index κ. This is equivalent to saying that X_n/n^κ converges in distribution to a related non-degenerate limit, which is not gaussian.

Here, we find analogous results for PRWSE (Theorem 6.1 below also gives the limit distribution of $\{X_n\}$, even when X_n/n has a positive limit). We introduce a related branching process in a random environment, which serves to estimate the time T_n, at which the random walk hits the site n. The main condition that we require on the environment is of "fluctuation type;" namely, we will assume that the distribution of $\log(\mu_0/\lambda_0)$ is not supported by an arithmetic progression $h\mathbb{Z}$ (with $h \in \mathbb{R}$).

Throughout this section, for simplicity we will state our results under the probability $P_0^{\pi,1}$, which in the sequel will be denoted by P. Here are some notations that will be used.

For $C > 0$ and $0 < \kappa < 2$ given constants, we will denote by $L_{\kappa, C}$ the "centered" stable distribution of index κ and parameter C (see ref. 5) whose characteristic function is as follows:

$$\text{If } 0 < \kappa < 2,\ \kappa \neq 1: \qquad \varphi_{\kappa, C}(t) = \exp\left\{-C\,|t|^\kappa \left(1 - i\frac{t}{|t|}\, tg\left(\frac{\pi}{2}\kappa\right)\right)\right\}$$

$$\text{If } \kappa = 1: \qquad \varphi_{\kappa, C}(t) = \exp\left\{-C\,|t| \left(1 + i\frac{t}{|t|}\frac{2}{\pi}\log t\right)\right\}$$

Let $\bar{L}_{\kappa, C}$ be the distribution defined by:

$$\text{If } 0 < \kappa < 1: \quad \bar{L}_{\kappa, C}(]-\infty, u]) = L_{\kappa, C}(]u^{-1/\kappa}, +\infty[)$$

$$\begin{aligned}\text{If } 1 \leqslant \kappa < 2: \quad \bar{L}_{\kappa, C}(]-\infty, u]) &= \bar{L}_{\kappa, C, A}(]-\infty, u]) \\ &= L_{\kappa, C}(]-uA^{1+1/\kappa}, +\infty[)\end{aligned}$$

for a given constant $A > 0$

Theorem 6.1. Let $\{(\lambda_j, \mu_j);\ j \in \mathbb{Z}\}$ be i.i.d. random variables satisfying (2.2) and such that

$$\prec \log \frac{\mu_0}{\lambda_0} \succ_\pi < 0 \tag{6.1}$$

there exists $0 < \kappa < +\infty$ such that

$$\prec \left(\frac{\mu_0}{\lambda_0}\right)^\kappa \succ_\pi = 1,$$

$$\prec \left(\frac{\mu_0}{\lambda_0}\right)^\kappa \log^+ \frac{\mu_0}{\lambda_0} \succ_\pi < +\infty, \qquad \prec \left(\frac{1-\lambda_0}{\lambda_0}\right)^\kappa \succ_\pi < +\infty \tag{6.2}$$

$$\text{the distribution of } \log(\mu_0/\lambda_0) \text{ is non-arithmetic.} \tag{6.3}$$

Then the following convergences in distribution hold with $A_\kappa > 0$, $B_i < +\infty$, $C > 0$ suitable constants:

(i) If $0 < \kappa < 1$,

$$\frac{T_n}{n^{1/\kappa}} \to L_{\kappa, C} \qquad \text{and} \qquad \frac{X_n}{n^\kappa} \to \bar{L}_{\kappa, C}$$

(ii) If $\kappa = 1$,

$$\frac{T_n - A_1 n D(n/l)}{n} \to L_{1, C} \qquad \text{and} \qquad \frac{X_n - \delta(n)}{n/\log^2 n} \to \bar{L}_{1, C}$$

where $l > 0$, $D(n) \sim \log n$ and $\delta(n) \sim n/(A_1 \log n)$

(iii) $1 < \kappa < 2$,

$$\frac{T_n - A_\kappa n}{n^{1/\kappa}} \to L_{\kappa, C} \qquad \text{and} \qquad \frac{X_n - A_\kappa^{-1} n}{n^{1/\kappa}} \to \bar{L}_{\kappa, C}$$

(iv) If $\kappa = 2$,

$$\frac{T_n - A_2 n}{B_1 \sqrt{n \log n}} \to \mathcal{N}(0, 1) \qquad \text{and} \qquad \frac{X_n - A_2^{-1} n}{A_2^{-3/2} B_1 \sqrt{n \log n}} \to \mathcal{N}(0, 1)$$

where $\mathcal{N}(0, 1)$ denotes the centered gaussian law with variance 1.

(v) If $\kappa > 2$,

$$\frac{T_n - B_3 n}{B_2 \sqrt{n}} \to \mathcal{N}(0, 1) \qquad \text{and} \qquad \frac{X_n - B_3^{-1} n}{B_3^{-3/2} B_2 \sqrt{n}} \to \mathcal{N}(0, 1)$$

Proof. The proof is an adaptation of that in ref. 9. Similarly to Section 3, for each site $n \in \mathbb{N}$, let us define for $j \in \mathbb{Z}$,

$$\begin{aligned} U_j^n &= \#\{k : 0 \leqslant k < T_n \text{ and } X_k = j,\ X_{k+1} = j-1\} \\ &= \text{the number of steps performed by the PRWSE} \\ &\quad \text{from } j \text{ to } j-1 \text{ before time } T_n \end{aligned}$$

We can write the analogue of (3.2) for T_n:

$$T_n = n + 2 \sum_{j=-\infty}^{n} U_j^n$$

Under (6.1), Theorem 2.1 shows that $X_n \to +\infty$ a.s. Thus $(1/n^{1/\kappa}) \times \sum_{j=-\infty}^{0} U_j^n \to 0$ a.s. and it suffices to show that the distribution of

$$\sum_{j=0}^{n} U_j^n \tag{6.4}$$

converges to $L_{\kappa, C}$ after a suitable normalization. To do so, we describe below a certain aspect of the distribution of (6.4).

When an environment $\mathscr{E}$ is fixed, Kesten *et al.*[9] showed (in the special case $\mu_j = 1 - \lambda_j$) that for a fixed $n \geqslant 1$, $U_n^n = 0, U_{n-1}^n, \ldots, U_1^n$ have the same law as the first n generations, $Z_0 = 0, Z_1, Z_2, \ldots, Z_{n-1}$, of a branching process with one immigrant at each generation. The immigrant arriving at time $n - j - 1$ corresponds to the first time the walker arrives at the site j (coming from the origin). In the present model, the unique modification (due to the fact that the condition $\mu_j = 1 - \lambda_j$ is relaxed) to Kesten, Kozlov, and Spitzer's description is that the offspring distribution of each particle present at time $n - j - 1$ is

$$P(B_j = k) = \begin{cases} 1 - \mu_j & \text{for} \quad k = 0 \\ \mu_j \lambda_j (1 - \lambda_j)^{k-1} & \text{for} \quad k \geqslant 1 \end{cases} \tag{6.5}$$

while the offspring distribution of the immigrant arriving at that time is

$$P(I_j = k) = \lambda_j (1 - \lambda_j)^k; \qquad k \geqslant 0 \tag{6.6}$$

(for the definitions concerning branching processes with immigration, see e.g. refs. 3 and 10).

When the environment is random (i.e., under probability P), since $(\mu_{n-1}, \lambda_{n-1}), \ldots, (\mu_1, \lambda_1)$ have the same joint distribution as $(\mu_0, \lambda_0), \ldots, (\mu_{n-2}, \lambda_{n-2})$, (6.4) has the same distribution as

$$\sum_{t=0}^{n-1} Z_t \tag{6.7}$$

where $\{Z_t\}_{t\geqslant 0}$ is a branching process in a random environment with one immigrant per generation and offspring distributions described by:

(6.5) for the individuals present at time j;

(6.6) for the immigrant arriving at time j.

Let us use the following representation.

$$Z_t = \sum_{s=0}^{t-1} Z_{s,t}; \qquad t \geqslant 0$$

where $Z_{s,t}$ stands for the number of progeny alive at time t of the immigrant who arrived at time s, $s<t$. We derive that

$$E(Z_{s,t+1} \mid Z_{s,t}=1, \mathscr{E}) = m_t, \qquad E((Z_{s,t+1})^2 \mid Z_{s,t}=1, \mathscr{E}) = m_t + 2m_t r_t$$

$$E(Z_{s,s+1} \mid \mathscr{E}) = r_t, \qquad E((Z_{s,s+1})^2 \mid \mathscr{E}) = r_s + 2r_s^2$$

From now, the proof of Theorem 6.1 will be the same as in ref. 9. For this reason, we will only outline the main arguments. Let $v_0 = 0$ and

$$v_n = \inf\{t > v_{n-1} : Z_t = 0\}, \qquad n \geqslant 1$$

the times at which the branching process extincts and restarts with the immigrant arriving at those times. Let us denote by

$$W_n = \sum_{t=v_n}^{v_{n+1}-1} Z_t$$

the total population produced between the times v_n and v_{n+1}. Thus, since the environment $\{(\lambda_j, \mu_j); j \in \mathbb{Z}\}$ is i.i.d., the random pairs

$$(v_{n+1} - v_n, W_n); \qquad n \geqslant 0$$

are also i.i.d.. The main argument is the following.

In the first place, we will show (see Lemma 6.2 below) that

$$l = E(v_{n+1} - v_n) < +\infty$$

so that (6.7) will satisfy

$$\sum_{i=0}^{n} Z_t \sim \sum_{i=0}^{n/l} W_i$$

The second member is a sum of i.i.d. random variables. The problem then is to investigate the properties of the distribution of the W_i's and next to use classical limit theorems for sums of i.i.d. random variables.

In the second place, under the assumptions of the theorem, one can show that W_0 is in the domain of attraction of a stable law with index κ. More precisely, we will show that for $C>0$ a suitable constant,

$$P(W_0 > x) \sim C x^{-\kappa}, \qquad x \to +\infty \tag{6.8}$$

To prove (6.8), note that the randomness of W_0 is due to the randomness of the environment on the one hand and to the fluctuations of the reproduction of each particle given the environment on the other hand. As in ref. 9, the tail of W_0 can be approximated by the tail of

$$\eta_\sigma(\mathscr{E}) . Z_\sigma$$

where

$\sigma = \sigma_A = \min\{t : Z_t > A\}$; with $A > 0$ sufficiently large,

$\eta_t(\mathscr{E}) = r_t \sum_{k=0}^{+\infty} m_{t+1} m_{t+2} \cdots m_{t+k}$.

Note that given σ, the r.v.'s $\eta_\sigma(\mathscr{E})$ and Z_σ are independent and $\eta_1, \eta_2,...$ depend only on the environment and have the same distribution as

$$\eta_0 = E\left(\sum_{t=1}^{+\infty} Z_{0,t} \,\middle|\, \mathscr{E} \right)$$

= the expected number of the total population, produced by the immigrant who arrived at time 0.

Next, according to Theorem 5 of ref. 8, we have

$$P\left(\left(\sum_{k=0}^{+\infty} m_1 m_2 \cdots m_k \right) > x \right) \sim K x^{-\kappa} \quad x \to +\infty, \quad K > 0 \text{ a suitable constant}$$

From the independence of r_0 and $\sum_{k=0}^{+\infty} m_1 m_2 \cdots m_k$, and the fact that $\prec (r_0)^\kappa \succ_\pi < +\infty$ (last part of (6.2)), it follows that

$$P(\eta_0 > x) \sim K \prec (r_0)^\kappa \succ_\pi x^{-\kappa} \qquad x \to +\infty \tag{6.9}$$

Formula (6.9) plays the role of (2.9) in ref. 9. From this, some relatively simple manipulations show that Lemmas 3, 4 and 5 in ref. 9 are still in force. This leads to (6.8) with

$$C = K \prec (r_0)^\kappa \succ_\pi \lim_{A \to +\infty} E((Z_{\sigma_A})^\kappa, \sigma_A < v_1) \tag{6.10}$$

The only part that is not a line by line rewriting of Kesten, Kozlov and Spitzer's proof is the Lemma 6.2 below (which is Lemma 2 of ref. 9).

Lemma 6.2. There exists two positive constants c_1 and c_2 such that

$$P(v_1 > t) < c_1 e^{-c_2 t}; \qquad t > 0$$

Proof. The lemma is a special case (nonrandom immigration) of Key's[10] Theorem 4.2. To see this, one can represent the process $\{Z_t\}$ as a component of a two-type branching process in a random environment, $\{(Z_t, I_t)\}$, with one type-I immigrant each unit of time. The individuals of each type (type-Z and type-I) only give birth to type-Z particles. The offspring distribution for type-Z individuals present at time j is given by (6.5) while the offspring distribution for type-I individuals present at time j including the immigrant is given by (6.6) (Note that $I_t = 0$, for all t). Note that (2.2), (6.1), (6.2) imply that assumptions (i), (ii) and (iii) of Theorem 4.2 in ref. 10 are satisfied. ∎

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